Example 10.3.20

Let $I=\{1,2\}$ $\mathcal{G}$ connected semi-Cartan rank 2

Assume $a_{12}^{Y}, a_{21}^{Y} \leq -2$, $\forall Y \in \mathcal{X}$

Then $\mathcal{G}$ is an infinite Cartan graph

$$X_1 \quad \begin{pmatrix} 2 & \boxed{-2} \\ \boxed{-2} & 2 \end{pmatrix} \qquad X_2 \quad \begin{pmatrix} 2 & \boxed{-2} \\ \boxed{-3} & 2 \end{pmatrix}$$

$r_1 = (12)$

$r_2 = id$

$$\underline{\underline{\Delta^{X_1 re}}}$$

$(X_1, 1) : (2, \boxed{3}, 2, 2)^{\infty}$

$(X_1, 2) : (2, 2, 3, 2)^{\infty}$

$$\Delta^{X re} = \{\pm \underset{(X_1,1)}{\boxed{\beta_k}}, \pm \tau \underset{(X_1,2)}{\boxed{\gamma_k}}\}$$

β_k: $\beta_1 : \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\beta_2 : \begin{pmatrix} \boxed{2} & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\beta_3 : \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix} \quad \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

$\beta_4 : \quad \cdots = \begin{pmatrix} \boxed{8} \\ 5 \end{pmatrix}$

$\beta_5 : \begin{pmatrix} 11 \\ 7 \end{pmatrix} \quad \beta_6 = \begin{pmatrix} \boxed{14} \\ 9 \end{pmatrix} \quad \beta_7 : \begin{pmatrix} \boxed{31} \\ 20 \end{pmatrix}$

$\beta_8 = \begin{pmatrix} \boxed{48} \\ 31 \end{pmatrix}$

$(2, 2, \cdots\cdots) \quad \boxed{\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}}$

P362 = 三行

all ≥ 2 $\quad a_{k+1} - a_k = \boxed{(c_k - 2) a_k} + \underline{a_k - a_{k-1}}$

$b_{k+1} - b_k = \boxed{(c_k - 2) b_k} + b_k - b_{k-1}$

$\delta = \alpha_1 + \alpha_2$

$\Delta^{X re} = \mathbb{Z}\delta + \boxed{\Delta^{X re}}$

$\underline{\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}} \quad \underline{(1, 4, 1, 4 \cdots)}$

$\underline{(4, 1, 4, 1 \cdots)}$

Example 10.3.22

Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a connected semi-Cartan graph of rank 2. $i, j \in I$ $i \neq j$.

Assume that $\underline{a_{ij}^{X} = a_{ji}^{X} = -1}$.

The only sequence in A^{+} with consecutive entries 1 is $\overline{(1, 1, 1)}$

Hence by Thm 10.3.21, $\mathcal{G}$ is a Cartan graph

$\Leftrightarrow$ $a_{ji}^{Y} = a_{ij}^{Y} = -1 \ \forall Y \in \mathcal{X}$.

$(r_2 r_1)^3 (X) = X$

Prof:

$$\begin{tikzcd} r_i(x) \arrow[r, bend left, "r_i"] & x \end{tikzcd}$$

$\begin{pmatrix} 2 & -4 \\ \cdot & 2 \end{pmatrix}$ $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$(1,1)$

$(1,1,1)$

$\Rightarrow$: $\mathcal{G}$ is a $\boxed{\text{finite}}$ Cartan graph

$\Rightarrow (c_1, \cdots, c_{m_{ij}^{r_i(x)}}) \in \mathcal{A}^+$

$\Rightarrow (c_1, \cdots, c_{m_{ij}^{r_i(x)}}) = (1,1,1)$

$\Rightarrow m_{ij}^{r_i(x)} = 3 = m_{ij}^{x} \Rightarrow (r_2 r_1)^3(x) = x$

$\Leftarrow$: Case 1: 3 is the smallest

Then $\bar{\omega} = 6n - \sum_{i=1}^{2n} c_i = 18 - 6 = 12 \mid 12$ $\quad (r_2 r_1)^n(x) = x$

$\Rightarrow 12n / \bar{\omega} = 3$

$\Rightarrow (1,1,1) \in \mathcal{A}^+ \Rightarrow \mathcal{G}$ is a finite Cartan graph.

Case 2:

$(r_2 r_1)(x) = x \quad n = 1$

$\bar{\omega} = 6n - \sum_{i=1}^{2n} c_i = 4 \mid 12$

$\Rightarrow 12n / \bar{\omega} = 3 \Rightarrow (1,1,1) \in \mathcal{A}^+$

$\Rightarrow \mathcal{G}$ is a finite Cartan graph.

Def:

For any $n \geq 1$, a set of non-zero vectors $v_1, \cdots, v_k \in \mathbb{Z}^n$, $k \geq 1$ is said to be relatively prime, if the elementary divisors of the matrix in $\mathbb{Z}^{n \times k}$ with columns $v_1, \cdots, v_k$ is unit.

$\beta_1, \beta_2 \in \mathbb{Z}^2$ $\quad |(\beta_1, \beta_2)| = 1$

列向量

$\Rightarrow \beta_1, \beta_2$ 互素

Cor (0.3.24)

Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ with $I = \{i, j\}$ be a finite Cartan graph of rank 2.

Let $X \in \mathcal{X}$, $(c_k)_{k \geq 1}$ be the characteristic sequence of $\mathcal{G}$ with respect to (X, i).

Let $q = m_{ij}^{X}$ then $(c_1, \cdots, c_q) \in \mathcal{A}^+$.

For all $1 \leq k \leq q-2$, let $i_k \in \{2, 3, \cdots, k+1\}$

s.t. $\underline{(c_1, \cdots, c_q)} = \underline{V_{i_{q-2}} \cdots V_{i_2} V_{i_1}}\, \underline{(0,0)}$

Let $\underline{(\beta_1, \cdots, \beta_q)}$ be the sequence of elts of $\mathbb{Z}^2$ arising from $\underline{(\alpha_1, \alpha_2)}$ by inserting successively at the i_k-th where $1 \leq k \leq q-2$, the sum of the elts at $i_k - 1$ and i_k.

$\underline{(\alpha_1, \alpha_2)} \to (\alpha_1, \underline{\alpha_1 + \alpha_2}, \underline{\alpha_2})$

$(\cancel{\alpha_1}\ \alpha_1, 2\alpha_1+\alpha_2, \alpha_1+\alpha_2, \alpha_2)$

① $\underline{\Delta_{\phi}^{X re} = \{\beta_1, \cdots, \beta_{q_e}\}}$

② For any $1 \le k < q_e$, the matrix

$(\beta_k, \beta_{k+1}) \in \mathbb{Z}^{2\times 2}$ has $|\ \ | = 1$

行列式 In particular β_k and β_{k+1} relatively prime.

③ $\forall\ 1 < k < q_e$, β_k is a sum of two relatively prime positive roots.

Proof: Assume $i=1, j=2$.

Since $\mathcal{G}$ is a finite Cartan graph,

$r_j r_i$ has finite order.

Then $(C_1, \cdots, C_{q_e}) \in \mathcal{A}^+$.

Let $(\beta_k)_{k\ge 1}$ be the root sequence of (X, i)

$|\Delta_{\phi}^{X re}| = m_{ij}^X = \underbrace{\bar{m}_{ij}^X}$

Now $\Delta_{\phi}^{X re} = \underline{\{\beta_1, \cdots, \beta_{q_e}\}}$

by Lem 9.2.7(1), Prop 10.3.18.

Lem 9.2.7(1): Cartan

K is X-reduced $\Leftrightarrow$ $\beta_{\phi}^{X,K}, \cdots, \beta_{\ell}^{X,K}$ are distinct

$\Leftrightarrow \Lambda^X(K) \subseteq \mathbb{N}_0^I$.

10.3.18: $\beta_{2k+1} = \mathrm{id}_X (s_1 s_2)^k (\alpha_1)$

$\beta_{2k+2} = \mathrm{id}_X (s_1 s_2)^k s_1 (\alpha_2)$

Note that $\underset{1\le k\le q_e}{\beta_k} = \underset{\Lambda^X(K)}{\underset{\varpi}{K}} = (\underbrace{1.2.1.2\cdots}_{m_{ij}^X = q_e})$

$\Rightarrow \boxed{\Delta_{\phi}^{X re}} = \underline{\{\beta_1, \cdots, \beta_{q_e}\}}$

$\bar{m}_{ij}^X = q_e$

$\underbrace{(1.2, 1.2, \cdots)}_{\bar{m}_{ij}^X 个}$ — 最长元

Then the corollary follows from 10.3.13

Let $\boxed{(C_1, \cdots, C_{n+1}) = V_i(C_1', \cdots, C_n')}$

Let $\beta_k' = \eta(C_1') \cdots \eta(C_{k-1}')(\alpha_1)$

$\beta_k = \eta(C_1) \cdots \eta(C_{k-1})(\alpha_1)$

Then $\beta_k = \beta_k'$ $1 \le k < i$

$\beta_k = \beta'_{k-1} \quad k > i$

$\underline{\beta_i} = \boxed{\beta'_{i-1} + \beta'_i} = \beta_{i-1} + \beta_{i+1}$

Claim: Any $\underline{\text{insert root}}$ is the sum of two relatively prime roots.

$(\alpha_1, \alpha_2) \qquad |(\alpha_1, \alpha_2)| = 1$

$(\underline{\alpha_1}, \underline{\alpha_1+\alpha_2}, \alpha_2)$

$|(\alpha_1, \alpha_1+\alpha_2)| = \left|\begin{smallmatrix}\cancel{\alpha_1+\alpha_2}\\ (\alpha_1,\alpha_2)\end{smallmatrix}\right| = 1$

☐

Remark 10.3.25:

Example 10.3.26

Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ in $\underline{9.1.2}$ $\underline{9.1.15}$

$I = \{1,2\}, \quad \mathcal{X} = \{X_1, X_2\}$

$r_1 = (12) \quad r_2 = \mathrm{id}$

$A^{X_1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \qquad A^{X_2} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$

The characteristic sequence of $\mathcal{G}$ with $(X_1, 1)$ is $(C_k)_{k \geq 1} = \underline{(1,4,1,3)}^{\infty}$

$\bar{\omega} = 6n - \sum_{i=1}^{2n} c_i = 12 - 9 = 3 \mid 12$

$12n / \bar{\omega} = 8 \qquad (c_1, \dots, c_8)$

$= (1,4,1,3,1,4,1,3)$

$= \underline{V_3} V_4 V_3 V_4 V_3 V_2 (0,0)$

$\Rightarrow (c_1, \dots, c_8) \in \mathcal{A}^+$.

$\Rightarrow \mathcal{G}$ is Cartan, $m_{ij}^{X} = |\Delta_+^{X\,re}| = 8$

$$\begin{tikzcd}
\{1,2\} \arrow[r, "V_2"] & \{1, 12, 2\} \arrow[d, "V_3"] \\
& \{1, 12, 12^2, 2\} \arrow[d, "V_4"] \\
& \{1, 12, 12^2, 12^3, 2\}
\end{tikzcd}$$

$= \{1, 12, 1^3 2^4, 1^2 2^3, 1^3 2^5, 12^2, 12^3, 2\} = \Delta_+^{X_1\,re}$

Cor 10.3.27

Cor 10.3.28:

Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a finite Cartan graph of rank 2.

Then $\underline{\exists X \in \mathcal{X}}$, $i, j \in I$, $i \neq j$ s.t one of the following hold:

①

②

③

④

⑤

Proof: We may assume $\mathcal{G}$ is connected.

If $m_{ij}^{X} = 2$ for all $X \in \mathcal{G}$

then $a_{ij}^X = a_{ji}^X = 0$, $\forall X$.

Otherwise $m_{ij}^X \geq 3$ $\forall X$.

Let $Y \in \mathcal{X}$, $i \in I$

Let $(c_k)_{k \geq 1}$ be the characteristic sequence of (Y, i)

Then $(c_1, \cdots, c_{m_{ij}^Y}) \in \mathcal{A}^+$.

By Prop 10.3.11, $\exists$ a subsequence

$(1,1)$, $(1,2,a)$, $(2,1,b)$

$(1,3,1,b)$ with $1 \leq a \leq 3$

$3 \leq b \leq 5$

of $(c_1, \cdots, c_{m_{ij}^Y})$ or its reversal.

Then the claim follows from Lemma 10.3.16.

If appear $(1,1)$

then must exist X s.t $a_{ij}^X = -1 = a_{ji}^X$

$$\begin{pmatrix} 2 & \boxed{-1} \\ & 2 \end{pmatrix} \xrightarrow{r_1} \underbrace{\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}}_{X}$$

$\underline{(1,2,a)}$

$$\begin{pmatrix} 2 & \\ -1 & 2 \end{pmatrix} \xrightarrow{r_2} \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \xrightarrow{r_1} \begin{pmatrix} 2 & -2 \\ -a & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ & 2 \end{pmatrix} \cdots$$

~~Thm 10.3.28~~

Thm 10.3.29:

(1) Let $n \geq 2$, $(c_1, \cdots, c_n) \in \mathcal{A}^+$

$\mathcal{X} = \{1, 2, \cdots, 2n\}$, $I = \{1, 2\}$

Define $r_1, r_2 : \mathcal{X} \to \mathcal{X}$

$r_1 = (12)(34) \cdots$ ~~(2n-1)~~ $(2n-1 \ \ 2n)$

$r_2 = (23)(45) \cdots (2n-2 \ \ 2n-1)(2n \ \ 1)$

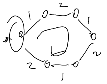
Then there is a unique semi-Cartan graph s.t the characteristic sequence of $(x = 1, 1)$ is $(c_1, \cdots, c_n)^\infty$

This $\mathcal{G}$ is a connected simply connected finite Cartan graph, and $\forall Y \in \mathcal{X}$

$m_{12}^Y = m_{21}^Y = n$

Proof: The existence of $\mathcal{G}$ is easy to check.

比如 $n=3$ $\mathcal{X}=\{1, \sim, 6\}$



$\begin{pmatrix} 2 & - \\ & 2 \end{pmatrix}$

Clearly, $\mathcal{G}$ is connected.

Moreover $(r_2 r_1)^{\boxed{k}}(x) \Leftrightarrow n \mid k$

Since $\sum_{i=1}^{n} c_i = 3n-6$ (Φ^+的个数)

$\bar{w} = 6n - \sum_{k=1}^{2n} c_k = 6n-(6n-12)$
$= 12 \mid 12$

$\Rightarrow \mathcal{G}$ is a Cartan graph and $m_{ij}^{X} = n$.

Finally, $\mathcal{G}$ is simply connected since $\forall X \in \mathcal{X}$, $\underline{\mathrm{Hom}(X,X) \text{ is generated by } \mathrm{id}_X (s_1 s_2)^n}$

$\underline{\mathrm{id}_X (s_1 s_2)^n = \mathrm{id}}$ by 9.2.22 (1)

(2) Any finite connected simply connected Cartan graph of rank 2 is iso to a Cartan graph (1) in.

Proof: Let $\mathcal{G}$ be ——.

$I = \{1, 2\}$.

Let n be the smallest n s.t. $(r_j r_i)^n(x) = x$

Since $\underline{\mathrm{id}_X (s_1 s_2)^{m_{ij}^X} = \mathrm{id}_X}$

we have $n \le m_{ij}^X$.

Let $(c_k)_{k \geq 1}$ —— $(X, 1)$

By the property of Cartan graph,

$F(\mathrm{id}_X (s_1 s_2)^k) \neq \mathrm{id}_{\mathbb{Z}^2}$ $0 < k < m_{ij}^X$

$F(\mathrm{id}_X (s_1 s_2)^k s_1) \neq \mathrm{id}_{\mathbb{Z}^2}$ $0 \le k < m_{ij}^X$

Since $\mathcal{G}$ is simply connected, this implies that the $2m_{12}^X$ points $\underline{r_1 (r_2 r_1)^k (X) \text{ and } (r_2 r_1)^k (X)}$, $0 \le k < m_{12}^X$ are pairwise distinct

Then $n = m_{12}^{x}$, $|x| = 2m_{12}^{x}$

since $\mathcal{G}$ is connected and rank 2,

Then $(C_1, \dots, C_n) \in \mathcal{A}^{+}$

$(C_k)_{k \geq 1} = (C_1, \dots, C_n)^{\infty}$

Thus (2) holds.

(3), Let $\mathcal{G}$ and $\mathcal{G}'$ be two semilinks as in (1) corresponding to:

$\underline{(C_1, \dots, C_q) \in \mathcal{A}^{+}}$

$\underline{(C_1', \dots, C_{q'}') \in \mathcal{A}^{+}}$

Then $\mathcal{G} \cong \mathcal{G}' \Leftrightarrow q = q'$ and

$(C_1, \dots, C_q)$, $(C_1', \dots, C_{q}')$

are in the same orbit of $\mathbb{D}_q \cdot \mathcal{A}^{+}(q)$

Proof: Clearly if $\mathcal{G} \cong \mathcal{G}'$,

then $q = q'$.

Then by construction, $\mathcal{G}' \overset{f}{\cong} \mathcal{G}$

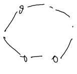
$\Leftrightarrow$ $\exists$ $X \in \{1, 2, \dots, 2n\}$ $i \in \{1, 2\}$

s.t. the characteristic sequence of $\mathcal{G}$ with respect to $(X, i)_{\in\{1,2\}}$

coincides with the ch— sequence of $\mathcal{G}'$ with respect to $(1, 1)$

Suppose $\gamma : I \to I = id$

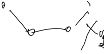
$X' = 1$



X'處 $\boxed{(1,1)}$

sequence

$f^{-1}(1) = X$



X處 $\boxed{(X,1)}$

$\boxed{(1,1)}$

(X, i)

(X, j) 是任何位置 X處 $\boxed{1013.16}$

Thus (3) follows from 1013.16.

Example 10.3.30

Let $\mathcal{G}$: $I = \{1, 2\}$

$$X = \{\underbrace{X_{\bar 0}, X_{\bar 1}}, \underbrace{X_{\bar 2}, X_{\bar 3}}\}$$

r_1 ~~0~~ $(X_{\bar 0}) = X_{\bar 1}$

$r_1(X_{\bar 2}) = X_{\bar 3}$

$r_2(X_{\bar 1}) = X_{\bar 2}$

$r_2(X_{\bar 0}) = X_{\bar 3}$

$A^{X_{\bar 0}} = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$ $A^{X_{\bar 1}} = \begin{pmatrix} 2 & -3 \\ -2 & 2 \end{pmatrix}$

$A^{X_{\bar 3}} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ $A^{X_{\bar 2}} = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$

The characteristic sequence of $(X_{\bar 0}, 1)$ is $(3, 2, 1, 3)^\infty$

The smallest n s.t $(r_2 r_1)^n (X_{\bar 0}) = X_{\bar 0}$ is $n = 2$.

Thus $\overline{\omega} = 3$ ~~$\overline{\omega} = 8$~~ , $\underline{\mathcal{G} \text{ is a finite Cartan graph.}}$

$m_{ij}^{X_{\bar 0}} = 8$.

$(3, 2, 1, 3, 3, 2, 1, 3) =$

$v_3 v_2 v_2 v_4 v_3 v_2 (0, 0) \in \mathcal{A}^+$

$\{1, 1^3 2, 1^5 2^2, 1^2 2, 12,$
$12^2, 12^3, 2\}$

We see that $1^5 2^2$, $1^{\text{[illegible]}} 2^2$ are roots,

but $\underline{1^3 2^2}$ is not a root.

However $(4\alpha_1 + 2\alpha_2)/2$ is a root

$(2\alpha_1 + 2\alpha_2)/2$ is a root

Prop 10.3.31

Let $\mathcal{G}$ be a finite Cartan graph of rank 2, let $a, b \in \mathbb{N}_0$

$b \geq 1$

Let $\underbrace{\underset{(b)}{a}\alpha_i + \cancel{b}\alpha_j \in \Delta_+^{X\mathrm{re}}}$

Then $c\alpha_i + \alpha_j \in \Delta_+^{X\mathrm{re}}$ $\forall\, 0 \leq c \leq \frac{a}{b}$.

Proof: By induction on $a+b$.

If $a+b=1$, then $a=0$, $b=1$ trivial.

Assume $a+b \geq 2$

By cor 10.3.27, the root ~~$\alpha\alpha$~~ $a\alpha_i + b\alpha_j$ is the sum of two positive roots

$\beta_1 = a_1\alpha_i + b_1\alpha_j$

$\beta_2 = a_2\alpha_i + b_2\alpha_j$ $a_i, b_i \in \mathbb{N}_0$

Case 1: Assume that $b_1 = 0$

then $\beta_1 = \alpha_i$

If $b=1$, then $\underbrace{a\alpha_i + \alpha_j \in \Delta_+^{X\mathrm{re}}}_{\text{by assumption.}}$

and $c\alpha_i + \alpha_j \in \Delta_+^{X\mathrm{re}}$ for $0 \leq c \leq a-1$

by induction hypothesis applied $\underbrace{\beta_2 = (a-1)\alpha_i + \alpha_j}_{b=1}$ ✓

If $b>1$,

then there's no int. c

with $\underbrace{\frac{a-1}{b} < c \leq \frac{a}{b}}_{\in \mathbb{N}_0}$

Here the claim holds by induction hypothesis applied to β_2 again.

Case 2: Assume $b_1, b_2 > 0$

Then $\underbrace{\frac{a_1}{b_1} \geq \frac{a}{b} \text{ or } \frac{a_2}{b_2} \geq \frac{a}{b}}_{(易证)}$

Thus the proposition holds by applying induction hypothesis to β_1 and β_2.

□